

SCHLÖMILCH AND BELL SERIES FOR BESSEL'S FUNCTIONS, WITH PROBABILISTIC APPLICATIONS.

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Abstracts.

We have introduced and investigated so-called Schlömilchs and Bell's series for modified Bessel's functions, namely, their asymptotic and non-asymptotic properties, connection with Stirling's and Bell's numbers etc.

We have obtained exact constants in the moment inequalities for sums of centered independent random variables, improved their asymptotical properties, found lower and upper bounds, calculated a more exact approximation, elaborated the numerical algorithm for their calculation, studied the class of smoothing, etc.

Keywords: Bessel's and Bell's function, Schlömilchs series and function, saddle-point method, Rosenthal's moment inequalities, Exact constants, Poisson distribution, Stirling's formula and numbers, Banach spaces of random variables.

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1 Introduction. Statement of problem.

Let $I_\nu(z)$ be usually **modified** Bessel's function:

$$I_\nu(z) = 2^{-\nu} z^\nu \sum_{k=0}^{\infty} 4^{-k} z^{2k} / (k! \Gamma(\nu + k + 1)), \quad (1.0)$$

where $\Gamma(\cdot)$ denotes the gamma-function and $\nu \geq 0$.

We define the so-called Schlömilch's functions F_3, F_2, F_1 of the first kind for the values $p \geq 0$, $\theta > 0$, $\beta > 0$:

$$F_3^{(S)} = F_3(p; \theta, \beta) = \sum_{k=-\infty}^{\infty} |k|^p \theta^k I_k(\beta), \quad (1.1a)$$

$$F_2(p; \beta) = F_3(p; 1, \beta) = \sum_{k=-\infty}^{\infty} |k|^p I_k(\beta), \quad (1.1b)$$

$$F_1(p) = F_3(p, 1, 1) = \sum_{k=-\infty}^{\infty} |k|^p I_k(1), \quad (1.1c)$$

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We also define the Schlömilch's functions *of a second kind* G_3, G_2, G_1 for positive integer values $p = 1, 2, 3, \dots$ only and $\theta > 0, \beta > 0$ as follows:

$$G_3^{(S)} = G_3(p; \theta, \beta) = \sum_{k=-\infty}^{\infty} k^p \theta^k I_k(\beta), \quad (1.1d)$$

$$G_2(p; \beta) = G_3(p, 1, \beta) = \sum_{k=-\infty}^{\infty} k^p I_k(\beta), \quad (1.1e)$$

$$G_1(p) = G_3(p, 1, 1) = \sum_{k=-\infty}^{\infty} k^p I_k(1). \quad (1.1f)$$

Of course, $G_3(2m; \theta, \beta) = F_3(2m; \theta, \beta)$, $m = 1, 2, 3, \dots$

Recall that $I_{-k}(z) = I_k(z)$, $k = 1, 2, 3, \dots$ Therefore,

$$F_3(p; \theta, \beta) = F_3(p; 1/\theta, \beta).$$

As for the definition in the case *usual*, i.e. *non-modified* Bessel's function and respective preliminary results see, for example, [1], [2], p. 38 - 42.

Let us examine the probabilistic meaning of these functions. Let ξ, η be independent random variables characterized by the Poisson distribution with parameters $\lambda > 0$ and $\mu > 0$ correspondently: $Law(\xi) = Poisson(\lambda)$, $Law(\eta) = Poisson(\mu)$, or, in more detail:

$$\mathbf{P}(\xi = k) = \exp(-\lambda) \lambda^k / k!, \quad k = 0, 1, 2, \dots;$$

$$\mathbf{P}(\eta = l) = \exp(-\mu) \mu^l / l!, \quad l = 0, 1, 2, \dots;$$

and denote $\tau = \xi - \eta$. Then we obtain for non-negative integer values n :

$$\mathbf{P}(\tau = n) = \sum_{k=0}^{\infty} \mathbf{P}(\xi = n + k) \mathbf{P}(\eta = k) =$$

$$\sum_{k=0}^{\infty} \exp(-\lambda - \mu) \lambda^{n+k} \mu^k / ((k+n)! n!) =$$

$$\exp(-(\lambda + \mu)) (\lambda/\mu)^{n/2} I_n \left(2 \sqrt{\lambda \mu} \right), \quad n = 0, 1, 2, \dots$$

and therefore

$$\begin{aligned} \mathbf{E}|\tau|^p &= \exp(-(\lambda + \mu)) \sum_{n=-\infty}^{\infty} |n|^p (\lambda/\mu)^{n/2} I_n \left(2 \sqrt{\lambda \mu} \right) = \\ &\exp(-(\lambda + \mu)) F_3 \left(p, \sqrt{\lambda/\mu}, 2 \sqrt{\lambda \mu} \right), \end{aligned} \quad (1.2)$$

$$\mathbf{E}\tau^p = \exp(-(\lambda + \mu)) \sum_{n=-\infty}^{\infty} n^p (\lambda/\mu)^{n/2} I_n(2 \sqrt{\lambda \mu}) =$$

$$\exp(-(\lambda + \mu)) G_3 \left(p, \sqrt{\lambda/\mu}, 2 \sqrt{\lambda \mu} \right). \quad (1.3)$$

(The case of negative integers n is considered in a similar way.)

If $\lambda = \mu$, then

$$\mathbf{E}|\tau|^p = \exp(-2\lambda) F_2(p, 2\lambda),$$

and in the case of $\lambda = \mu = 1/2$: we obtain:

$$\mathbf{E}|\tau|^p = F_1(p)/e.$$

Here we introduce some new functions which represent generalizations of the classical Bell's number and functions.

Generalized Bell functions *of a first kind* $B_4(p; a, \lambda, \gamma)$ can be defined as

$$B_4(p; a, \lambda, \gamma) = \sum_{n=0}^{\infty} \frac{|n-a|^p \lambda^n}{\exp(\lambda) \cdot \Gamma(n + \gamma + 1)}, \quad (1.3a)$$

where $\lambda > 0, \gamma \geq 0$. We also define

$$B_3(p; a, \lambda) = B_4(p; a, \lambda, 0) = \sum_{n=0}^{\infty} \frac{|n-a|^p \lambda^n}{\exp(\lambda) n!}, \quad (1.3b)$$

$$B_2(p; \lambda) = B_3(p; 0, \lambda); \quad (1.3c)$$

$$B(p) = B_1(p) = B_2(p; 1) = \sum_{n=0}^{\infty} \frac{n^p}{e n!}. \quad (1.3d)$$

We also define generalized Bell functions *of a second kind* $D_4(p; a, \lambda, \gamma)$ for integer positive p values as follows:

$$D_4(p; a, \lambda, \gamma) = \sum_{n=0}^{\infty} \frac{(n-a)^p \lambda^n}{\exp(\lambda) \cdot \Gamma(n + \gamma + 1)}, \quad (1.4a)$$

$\lambda > 0, \gamma \geq 0$. We also define $D_3(p; a, \lambda) = D_4(p; a, \lambda, 0)$; $D_2(p; \lambda) = D_3(p; 0, \lambda)$; $D_1(p) = D_2(p; 1) = B_1(p)$.

The probabilistic interpretation is

$$\mathbf{E}|\xi - a|^p = e^{-\lambda} \sum_{n=0}^{\infty} |n-a|^p \lambda^n / n! = B_3(p; a, \lambda)$$

and $\mathbf{E}(\xi - a)^p = D_3(p; a, \lambda)$.

In this paper we investigate the estimations and asymptotic for introduced functions as $p \rightarrow \infty$.

In Section 4 we find exact constants for moment estimations of sums of independent random variables.

2 Non-asymptotic properties. Relations with Stirling's numbers.

Recall that the Stirling numbers of the second kind $\{s(n, r)\}$ which have appeared in the combinatorial theory ([3], p. 117, [4], p. 234 - 243), are defined by the following identities:

$$x^n = \sum_{r=0}^n s(n, r) x_{(r)}; x_{(r)} = x(x-1)(x-2) \dots (x-r+1), \quad x_{(0)} = 1.$$

1. Define the function $Q_{2m}(\lambda, \mu)$ for the values $m = 1, 2, 3, \dots$, $\lambda, \mu > 0$ by the following way:

$$Q_{2m}(\lambda, \mu) \stackrel{def}{=} \exp(-\lambda - \mu) F_3 \left(2m; \sqrt{\lambda/\mu}, 2\sqrt{\lambda\mu} \right).$$

Proposition:

$$Q_{2m}(\lambda, \mu) = \sum_{r=0}^{2m} \sum_{i=0}^r \sum_{j=0}^{2m-r} (-1)^r \binom{2m}{r} \lambda^i \mu^j s(r, i) s(2m-r, j).$$

Therefore, the function $Q_{2m}(\lambda, \mu)$ is a polynomial of the power $2m$.

Proof. As known $Q_{2m}(\lambda, \mu) = \mathbf{E}(\xi - \eta)^{2m}$. The conclusion 1 follows from the definition of Stirling's numbers, binomial formula and the following identity

$$\mathbf{E}\xi_{(r)} = \lambda^r.$$

2. Let us consider the function $D_3(p; a, \lambda)$ for integer non-negative values p . We have:

$$\begin{aligned} D_3(p; a, \lambda) &= \mathbf{E}(\xi - a)^p = \sum_{l=0}^p (-1)^l \binom{p}{l} a^{p-l} \sum_{r=0}^l \lambda^r s(l, r) = \\ &= \sum_{r=0}^p \lambda^r \sum_{l=r}^p (-1)^l \binom{p}{l} a^{p-l} s(l, r). \end{aligned}$$

3. Let us denote for integer positive p the values:

$$H(p; \lambda, \mu) = \exp(-\lambda - \mu) G_3(p; \sqrt{\lambda/\mu}, 2\sqrt{\lambda\mu}),$$

where $\lambda = \lambda_1 + \lambda_2$, $\mu = \mu_1 + \mu_2$, $\lambda_{1,2} > 0$, $\mu_{1,2} > 0$. We assert that

$$H(p; \lambda_1 + \lambda_2, \mu_1 + \mu_2) = \sum_{r=0}^p \binom{p}{r} H(r; \lambda_1, \mu_1) H(p-r; \lambda_2, \mu_2).$$

Namely, we can represent (on some Probability space) the random variables ξ and η as a sums $\xi = \xi_1 + \xi_2$, $\eta = \eta_1 + \eta_2$, where all the variables are independent, $Law(\xi_i) = Poisson(\lambda_i)$, $Law(\eta_i) = Poisson(\mu_i)$, $i = 1, 2$; then

$$H(p; \lambda_1 + \lambda_2, \mu_1 + \mu_2) = \mathbf{E}((\xi_1 - \eta_1) + (\xi_2 - \eta_2))^p = \sum_{r=0}^p \binom{p}{r} \mathbf{E}(\xi_1 - \eta_1)^r \mathbf{E}(\xi_2 - \eta_2)^{p-r} =$$

$$\sum_{r=0}^p \binom{p}{r} H(r; \lambda_1, \mu_1) H(p-r, \lambda_2, \mu_2).$$

3 Asymptotic results.

Denote $q = q(p; a, \gamma, \lambda) = (p - a - \gamma - 0.5)/\lambda$,

$$p \geq 4 \Rightarrow g(p) = \frac{p}{e \log p}, \Delta(p) = \frac{\log \log p}{\log p},$$

and assume $a, \gamma, \lambda = \text{const}$, $p \rightarrow \infty$.

Let us introduce the following relations of equivalence: for two positive functions $f_1(p), f_2(p)$ we will write $f_1 \sim f_2$ iff

$$f_1(p) = f_2(p)(1 + O(1/\log p)).$$

We can also write $f_1(\cdot) \asymp f_2(\cdot)$, iff $f_1^{1/p}(p)/g(p) \sim f_2^{1/p}(p)/g(p)$.

Note that if $f(p) = C \cdot h(p)$, $C = \text{const} > 0$, then $f^{1/p}(p) \sim h^{1/p}(p)$.

Theorem 3.1.

$$\mathbf{a.} \quad \lambda^{-1} B_4^{1/q}(p; a, \lambda, \gamma)/g(q) \sim 1 + \Delta(q). \quad (3.1)$$

b. Let us denote $\Lambda = \beta \max(\theta, 1/\theta)$ for the function $F = F_3(p; \theta, \beta)$. We assert that:

$$F_3(p; \theta, \beta) \asymp B_4(p; 0, \Lambda, 0); \quad (3.2)$$

$$\mathbf{c.} \quad F_3(p; \theta, \beta) \asymp G_3(p; \theta, \beta); \quad (3.3)$$

$$\mathbf{d.} \quad D_4(p; a, \lambda, \gamma) \asymp B_4(p; a, \lambda, \gamma); \quad (3.4)$$

Proof. STEP 1. We assert that $\lambda > 0 \Rightarrow$

$$\frac{\lambda^n}{n!} \leq I_n(2\sqrt{\lambda}) \leq \frac{\lambda^n}{n!} \cdot \frac{\exp(\lambda) - 1}{\lambda}. \quad (3.5)$$

The first inequality is evident; let us prove the second one. We have:

$$\lambda^{-n} I_n(2\sqrt{\lambda}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k! (n+k+1)!} =$$

$$\frac{1}{0!(n+1)!} + \frac{\lambda}{1!(n+2)!} + \frac{\lambda^2}{2!(n+3)!} + \dots \leq$$

$$\frac{1}{n!} \left(1 + \frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \dots \right) = \frac{1}{n!} \frac{\exp(\lambda) - 1}{\lambda}.$$

STEP 2.

$$B_4(p; a, \lambda, \gamma) \asymp \lambda^a B_3(p; \lambda, a + \gamma).$$

Proof. We can assume without loss of generality that a is integer. Further,

$$B_4(p; a, \lambda, \gamma) \asymp \sum_{k>a} \frac{(k-a)^p \lambda^k}{\Gamma(k+\gamma+1)} = \lambda^a \sum_{n=0}^{\infty} \frac{n^p \lambda^n}{\Gamma(k+a+\gamma+1)}.$$

Let us denote for $\alpha > 0$ the function

$$A(\alpha) = (\alpha + 1)^{\alpha+0.5} e^{2\alpha+1/12}.$$

It follows from Stirling's formula that at $x \geq 1$

$$1 \leq \frac{\Gamma(x+\alpha)}{\Gamma(x) \cdot x^\alpha} \leq A(\alpha).$$

Hence

$$A(\alpha) B_4(p; a, \lambda, \gamma) \asymp \sum_{m=1}^{\infty} \frac{m^{p-a-\gamma} \lambda^m}{m!} = B_4(p-a-\gamma; 0, \lambda, 0).$$

Thus, we have obtained a simple case $B_2(\cdot, \cdot)$ instead of $B_4(\cdot)$.

STEP 3. We consider in this section the function $F_3(p; \theta, \beta)$. We can suppose that $\theta \geq 1$. Denote

$$F^+(p; \theta, \beta) = \sum_{k=0}^{\infty} k^p \theta^k I_k(\beta).$$

It follows, by virtue of the bide-side inequality $F^+ \leq F_3 \leq 2F^+$ that $F_3 \asymp F^+$.

Furthermore, we obtain using (3.5)

$$F^+(p; \theta, \beta) \asymp \sum_{k=1}^{\infty} \frac{k^p (\theta\beta)^k}{k!} \asymp B_4(p; 0, \Lambda, 0) = B_2(p; \Lambda).$$

STEP 4. Instead of the general case, we need to investigate only the case B_2 . We obtain using the Stirling formula with remainder term again, $B_2 = B_2(p; \lambda) =$

$$\sum_{m=1}^{\infty} \frac{m^p \lambda^m}{m!} \asymp \sum_{m=2}^{\infty} \exp(-m \log m + m \log(e\lambda) + (p-0.5)).$$

Replacing the last sum by an integral, we can see that

$$B_2 \asymp \int_2^{\infty} \exp(U(p; x)) dx \stackrel{\text{def}}{=} V(p) = V.$$

$$U(p; x) = -x \log x + x \log(e\lambda) + (p-1/2) \log x.$$

An integral for V can be estimated by means of classical saddle-point method [4], p. 119 - 127.

In the next section we consider more exact estimations with non-asymptotic results in the particular case.

4 Probabilistic applications and non-asymptotic results.

Let $p = \text{const} \geq 2$, $\{\xi(i)\}$, $i = 1, 2, \dots, n$ be a sequence of independent centered $\mathbf{E}\xi(i) = 0$ random variables belonging to the space L_p , i.e. such that

$$\forall i \quad \|\xi(i)\|_p \stackrel{\text{def}}{=} \mathbf{E}^{1/p} |\xi(i)|^p < \infty. \quad (4.0)$$

We denote $\sum a(i) = \sum_{i=1}^n a(i)$, $L(p) = C^p(p)$, where

$$C(p) = \sup_{\{\xi(i)\}} \sup_n \frac{\|\sum \xi(i)\|_p}{\max \left(\|\sum \xi(i)\|_2, (\sum \|\xi(i)\|_p^p)^{1/p} \right)}, \quad (4.1)$$

where " C " denotes the centered case, and the external " \sup " is calculated over all the sequences of independent centered random variables that satisfy the condition (4.0).

If all the variables $\{\xi(i)\}$ in (4.0) are symmetrically distributed, independent, and $\|\xi(i)\|_p < \infty$, we denote the corresponding constants (more exactly, functions of p) $S(p)$ (" S " denoting the symmetrical case) instead of $C(p)$ and $K(p) \stackrel{\text{def}}{=} S^p(p)$ instead of $L(p)$. Obviously, $S(p) \leq C(p)$, $K(p) \leq L(p)$. It is proved in [5] that $C(p) \leq 2S(p)$, $L(p) \leq 2^p K(p)$.

The constant $C(p)$, $S(p)$ are called the exact constants in the moment inequalities for the sums of independent random variables. They play very important role in the classical probability theory ([6], 522 - 523, [7], p. 63;), in the probability theory on the Banach spaces ([8], [9], in the statistics and theory of Monte - Carlo method ([10], section 5) etc.

There are many publications on the behavior of the constants $C(p)$, $S(p)$ as $p \rightarrow \infty$. The first estimations were obtained in [11]; Rosenthal [12] proved, in fact, that $C(p) \leq C_1^p$; $C_1 = \text{const} > 1$; here and after C_j , $j = 1, 2, \dots$ are positive finite *absolute* constants. It is proved in [5], [13] that $C(p) \leq 9.6 p / \log p$. In the works [14], [15] are obtained the *non - asymptotic* bid-side estimations for $S(p)$:

$$(e\sqrt{2})^{-1} p / \log p \leq S(p) \leq 7.35 p / \log p, \quad p \geq 2, \quad (4.2)$$

and there are some moment estimations for the sums independent non-negative random variables. See also Latala [5], Utev [16], [17]; Pinelis and Utev [17] etc.

Ibragimov R. and Sharachmedov Sh. [8], [9] and Utev [17], [16] have obtained the *explicit* formula for $S(p)$: $S(2) = 1$; at $p \in (2, 4]$

$$S(p) = \left(1 + \sqrt{\frac{2^p}{\pi}} \Gamma\left(\frac{p+1}{2}\right) \right)^{1/p};$$

$$p \geq 4 \Rightarrow S(p) = \|\tau_1 - \tau_2\|_p, \quad (4.3)$$

where a random variables τ_j are independent and have the Poisson distribution with parameters equal to 0.5: $\mathbf{E}\tau_j = \mathbf{Var} \tau_j = 1/2$.

As a consequence, it was obtained that as $p \rightarrow \infty$

$$S(p) = \frac{p}{e \cdot \log p} \left(1 + o \left(\frac{\log^2 \log p}{\log p} \right) \right).$$

The following representation for the values $L(2m), m = 2, 3, 4, \dots$: is obtained in [19]:

$$C^{2m}(2m) = L(2m) = \mathbf{E}(\theta - 1)^{2m} = e^{-1} \sum_{n=0}^{\infty} (n - 1)^{2m} / n! = B_1(p),$$

where the random variable θ has the Poisson distribution with parameter 1, and there is a hypothesis that for all values $p \geq 4$ $C^p(p) = L(p)$.

Here we denote

$$L(p) = \mathbf{E}|\theta - 1|^p = e^{-1} \sum_{n=0}^{\infty} |n - 1|^p / n! = B_4(p; 1, 1, 0), \quad G(p) = L^{1/p}(p). \quad (4.4)$$

for all the values $p \geq 4$. In [20] estimations for $C(p)$ are obtained in the case where the sequence $\{\xi(i)\}$ is a sequence of martingale - differences, in [9], [8] are presented some generalizations for weakly dependent random variables $\{\xi(i)\}$.

In this section we have improved the known bid-side estimations and asymptotic for $S(p)$, $G(p)$ at $p \rightarrow \infty$, found the exact boundaries for the different approximations of $S(p)$, $G(p)$; describe the algorithm for the numerical calculation of $K(p)$, $L(p)$; studied the analytical properties of $K(p)$, $L(p)$ etc.

Note that there are many other statements of this problem: for the non-negative variables [12], [14]; for the Hilbert space valued variables ([8], [16]) etc.

Let us denote at $p \geq 4$: $\delta(p) = 1/\log p$,

$$h(p) = g(p) \left(1 + \Delta(p) + \Delta^2(p) \right) =$$

$$\left[p / (e \log p) \right] \cdot \left(1 + \log \log p / \log p + (\log \log p / \log p)^2 \right);$$

Theorem 4.1.

$$1 = \inf_{p \geq 4} G(p)/g(p) < \sup_{p \geq 4} G(p)/g(p) = C_3, \quad (4.5a)$$

where

$$C_3 = \sup_{p \geq 4} B_2^{1/p}(p; 1)/g(p) = G(C_4)/g(C_4) \approx 1.77638,$$

$$C_4 = \operatorname{argmax}_{p \geq 4} B_1^{1/p}(p)/g(p) \approx 33.4610;$$

(The equality $C_3 \approx 1.77638$ means that $|C_3 - 1.77638| \leq 5 \cdot 10^{-6}$);

$$1 = \inf_{p=4,6,8,\dots} C(p)/g(p) < \sup_{p=4,6,8,\dots} C(p)/g(p) = C_5, \quad (4.5b)$$

where $C_5 =$

$$\inf_{p=4,6,8,\dots} B_2^{1/p}(p; 1)/g(p) = G(C_6)/g(C_6) \approx 1.77637, \quad C_6 = 34;$$

$$1 = \inf_{p \geq 15} G(p)/h(p) < \sup_{p \geq 15} G(p)/h(p) = G(C_8)/h(C_8) = C_7, \quad (4.5c)$$

where

$$C_7 = \sup_{p \geq 15} B_2^{1/p}(p; 1)/h(p) \approx 1.2054,$$

$$C_8 = \operatorname{argmax}_{p \geq 15} B_2^{1/p}(p; 1)/h(p) \approx 71.430;$$

$$1 = \inf_{p \geq 4} S(p)/g(p) < \sup_{p \geq 4} S(p)/g(p) = C_9, \quad (4.6a)$$

where

$$C_9 = \sup_{p \geq 4} W^{1/p}(p)/g(p) = S(C_{10})/g(C_{10}) \approx 1.53572,$$

$$C_{10} = \operatorname{argmax}_{p \geq 4} F_1^{1/p}(p)/g(p) \approx 22.311;$$

$$1 = \inf_{p \geq 15} S(p)/h(p) < \sup_{p \geq 15} S(p)/h(p) = S(C_{12})/h(C_{12}) = C_{11}, \quad (4.6b)$$

where

$$C_{11} = \sup_{p \geq 15} F_1^{1/p}(p)/h(p) \approx 1.03734,$$

$$C_{12} = \operatorname{argmax}_{p \geq 15} F_1^{1/p}(p)/h(p) \approx 138.149;$$

$$1 = \inf_{p=16,18,20,\dots} C(p)/h(p) < \sup_{p=16,18,20,\dots} C(p)/h(p) =$$

$$C(72)/h(72) = \sup_{p=16,18,20,\dots} F_1^{1/p}(p; 1)/h(p) \approx 1.2053. \quad (4.6c)$$

(We have choosen the value 15 as long as the function $\log \log p / \log p$ monotonically decreases at the values $p \geq \exp(e) \approx 15.15426$).

Note that our estimations and constants (4.5a, 4.5b, 4.5c) and (4.6a, 4.6b, 4.6c) are exact and improve the constants and estimations of Rosenthal [12]; Johnson, Schechtman, Zinn et al. [14], [15]; Ibragimov, Sharachmedov [18],[19]; Latala [5]; Utev [16], [17] etc. For example, $1/(1/\sqrt{2}) \approx 1.41421$, $7.35e/C_3 \approx 11.2472$.

Theorem 4.2. *At $p \rightarrow \infty$ $G(p) = [p/(e \cdot \log p)] \times$*

$$\left(1 + \frac{\log \log p}{\log p} + \frac{1}{\log p} + \frac{\log^2 \log p}{\log^2 p} + \frac{\log \log p}{\log^2 p}(1 + o(1))\right); \quad (4.7a)$$

$$S(p) = [p/(e \cdot \log p)] \times$$

$$\left(1 + \frac{\log \log p}{\log p} + \frac{1 - \log 2}{\log p} + \frac{\log^2 \log p}{\log^2 p} + o\left(\frac{\log \log p}{\log^2 p}\right)\right). \quad (4.7b)$$

Let us denote by $N = N(p)$, $M = M(p)$ the solutions of equations, which are unique:

$$M(p) \log M(p) = p, \quad N(p) \log(2N(p)) = p, \quad (4.8)$$

for the values $p \geq 4$ such that $N(p) = 0.5M(2p)$.

Theorem 4.3. *At $p \rightarrow \infty$, $m = \text{const} = 2, 3, 4, \dots$, $m \rightarrow \infty$*

$$G(p) = M(p)^{1-M(p)/p} \exp(M(p)/p) (1 + O(\log p/p)), \quad (4.9a)$$

$$C(2m) = M^{1-M(2m)/(2m)}(2m) \exp(M(2m)/(2m)) (1 + O(\log m/m)),$$

$$S(p) = N(p) (e/(2N(p)))^{N(p)/p} (1 + O(\log p/p)). \quad (4.9b).$$

Theorem 4.4. *Let p be even: $p = 2m$, $m = 2, 3, 4, \dots$. Then*

$$K(2m) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} \sum_{q=0}^{2m-l} \sum_{r=0}^l 2^{-r-q} s(2m-l, q) s(l, r), \quad (4.10a)$$

$$C^{2m}(2m) = L(2m) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} \sum_{r=0}^{2m-l} s(2m-l, r). \quad (4.10b)$$

For integer odds values $p = 5, 7, 9, \dots$ we obtain the representation

$$G^p(p) = L(p) = (2/e) + \sum_{k=0}^p (-1)^k \binom{p}{k} F_1(p-k). \quad (4.11)$$

Proof. First, we consider some auxiliary results.

1. In the symmetrical case, for all the values $p \in [4, \infty)$ we have:

$$K(p) = (2/e) \sum_{n=1}^{\infty} n^p I_n(1) = F_1^{(S)}(p). \quad (4.12)$$

Namely, we obtain from (4.3) for the values τ_1, τ_2 at $n = 1, 2, \dots$:

$$\mathbf{P}(\tau_1 - \tau_2 = n) = e^{-1} \sum_{k=0}^{\infty} \frac{2^{-k} 2^{-(n+k)}}{k! (k+n)!} = I_n(1)/e.$$

2. On the basis of the equality (4.12) we can offer the numerical algorithm for $K(p)$ investigation, calculation and estimation. To improve convergence rate of series (4.12) we can write:

$$2\pi I_n(1) = \int_{-\pi}^{\pi} \exp(\cos(\theta)) \cos(n\theta) d\theta,$$

(see, for example, [21], p. 958, formula 5.) After the integration by parts, we obtain

$$2\pi I_n(1) = (-1)^m n^{-2m} \int_{-\pi}^{\pi} (\exp(\cos \theta))^{(2m)} \cos(n\theta) d\theta,$$

where $m = 1, 2, \dots$. Using the method of mathematical induction, we conclude:

$$(\exp \cos(\theta))^{(2m)} = \exp(\cos(\theta)) P_{2m}(\cos(\theta)),$$

where $P_{2m}(x)$ are polynomials of the degree $2m$, which can be calculated by recursion

$$P_{2m+2}(x) = (1 - x^2) \left(P_{2m}'' + 2P_{2m}'(x) + P_{2m}(x) \right) - x \left(P_{2m}'(x) + P_{2m}(x) \right)$$

with the initial condition $P_0(x) = 1$. Therefore, we obtain the following representation for $K(p)$:

$$\pi e K(p) = \sum_{n=1}^{\infty} n^{p-2m} \int_{-\pi}^{\pi} \exp \cos(\theta) P_{2m}(\theta) \cos(n\theta) d\theta. \quad (4.13)$$

3. Corollary. For even numbers $p = 2m$, $m = 1, 2, 3, \dots$ all the numbers $K(p) = K(2m)$, $L(p) = L(2m)$ are integer.

In fact, it follows from Equation (4.12) that

$$K(2m) = (\pi e)^{-1} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} g^{(2m)}(\theta) \cos(n\theta) d\theta =$$

$$e^{-1} (\exp(\cos \theta))^{(2m)}(0) = (-1)^m P_{2m}(1).$$

It is easy to verify that all the coefficients of polynomials $P_{2m}(x)$ are integer; thus, the number $P_{2m}(1)$ is integer.

The second conclusion of our corollary follows from Equation (4.10), as long as all the Stirling's numbers are integer.

4. For example, $K(6) = 31$, $L(6) = 41$. For non - integer values p we can use the method described above. We have obtained using a computer program:

TABLE 1

p	K(p)	L(p)	p	K(p)	L(p)
2	1	1	10.5	14000.4	41385.2
4	4	4	11	30403.2	98253.7
4.5	6.3358	6.6712	11.5	67091.3	236982
5	10.4118	11.7358	12	150349	580317
5.5	17.686	21.538	12.5	341951.2	1.44191E+006
6	31	41	13	788891.0	3.63328E+006
6.5	55.819	80.5508	13.5	1.84518E+006	9.27951E+006
7	103.22	162.7358	14	4.37346E+006	2.40112E+007
7.5	192.45	337.176	14.5	1.04998E+007	6.29176E+007
8	379	715	15	2.55231E+007	1.66888E+008
8.5	757.7	1549.28	15.5	6.27927E+007	4.47926E+008
9	1126.5	3425.7358	16	1.56298E+008	1.21607 E + 009
9.5	3015.0	7721.29	16.5	3.93475E+008	3.33839E+009
10	6556	17722	17	1.00153E+009	9.26407E+009

TABLE 2

p	K(p)	L(p)
17.5	2.57666E+009	2.59791E+010
18	6.69849E+009	7.36008E+010
18.5	1.75916E+010	2.106E + 011
19	4.66582E+010	6.08476 + 011
19.5	1.24952E+011	1.77473E+012
20	3.37789E+011	5.22427E+012
20.5	9.21603E+011	1.55177E+013
21	2.53714E+012	4.64999E+013

5. Using a discrete analog of the saddle - point method ([4], p. 262 - 264), [18]), we find that

$$M(p) = [p/\log p] \cdot (1 + \varepsilon(p)),$$

where at $p \rightarrow \infty$

$$\varepsilon(p) = \Delta(p) + \Delta^2(p) - \delta(p) \Delta(p) (1 + o(1)). \quad (4.14)$$

Hence

$$N(p) = [p/\log(2p)] \cdot (1 + \varepsilon(2p)) =$$

$$[p/\log p] \cdot \left[1 + \Delta(p) + \Delta^2(p) - \delta(p) \Delta(p) (1 + \log 2)(1 + o(1)) \right].$$

We define the following functions and constants for the values $p \geq P_0 = 700$:

$$C_{14} = (1 - \log \log P_0 / \log P_0)^{-1} \approx 1.402365,$$

$$C_{15} = 2 \cdot \left[(1 + 4\Delta^2(P_0))^{1/2} + 1 \right]^{-1} \approx 0.928958,$$

$$\zeta(p) = \log 2 / \log(2p),$$

$$\varepsilon_+(p) = \Delta(p) + C_{14}\Delta^2(p), \quad \varepsilon_-(p) = \Delta(p) + C_{15}\Delta^2(p),$$

$$M_+ = M_+(p) = [p/\log p] \cdot (1 + \varepsilon_+(p)),$$

$$M_- = M_-(p) = [p/\log p] \cdot (1 + \varepsilon_-(p)), \quad (4.15a)$$

$$N_+(p) = [p/(e \cdot \log(2p))] \cdot (1 + \varepsilon_+(2p)),$$

$$N_-(p) = [p/(e \cdot \log(2p))] \cdot (1 + \varepsilon_-(2p)). \quad (4.15b)$$

More exact calculation show us that for all the values $p \geq P_0$

$$M_-(p) \leq M(p) \leq M_+(p), \quad N_-(p) \leq N(p) \leq N_+(p).$$

Namely, one can readily see that $\forall p \geq P_0 \Rightarrow$

$$M_- \log M_- < p = M \log M < M_+ \log M_+.$$

6. Let us denote

$$b_1(x, p) = x^p / \Gamma(x + 1), \quad b_2(x, p) = x^p / (2^x \Gamma(x + 1));$$

$$V(x, p) = p \log x - x \log x + x,$$

$$X(p) = V(M(p), p) / p = \sup_{x \geq 4} V(x, p) / p,$$

$$W(x, p) = p \log x - x \log x + x(1 - \log 2),$$

$$Y(p) = W(N(p), p) / p = \sup_{x \geq 4} W(x, p) / p.$$

We have using the equality (4.14): $X(p) = \log[p/(e \cdot \log p)] +$

$$\begin{aligned} & \Delta(p) + \delta(p) + [\log(1 + \varepsilon(p)) - \varepsilon(p) + \Delta(p)\varepsilon(p)] + \\ & \{\delta(p)(\varepsilon(p) - \log(1 + \varepsilon(p))) - \delta(p)\varepsilon(p) \log(1 + \varepsilon(p))\} = \\ & \log[p/(e \cdot \log p)] + X_0(p), \end{aligned}$$

where at $p \geq P_0$ $X_2(p) < X_0(p) < X_1(p)$, $X_1(p) \stackrel{def}{=}$

$$\Delta(p) + \delta(p) + \Delta(p)\varepsilon_+(p) + \delta(p)[\varepsilon_+(p) - \log(1 + \varepsilon_+(p))], \quad (4.16a)$$

$$\begin{aligned} X_2(p) & \stackrel{def}{=} \Delta(p) + \delta(p) + [\log(1 + \varepsilon_-(p)) - \varepsilon_-(p)] - \\ & - \delta(p)\varepsilon_-(p) \log(1 + \varepsilon_-(p)). \end{aligned} \quad (4.16b)$$

The function $p \rightarrow X_1(p)$, $p \in [P_0, \infty)$ is monotonically decreasing and

$$\exp(X_1(P_0)) < 1.7563, \quad \lim_{p \rightarrow \infty} X_1(p) = 0. \quad (4.16c)$$

In the same manner we obtain: $Y(p) = \log[p/(e \cdot \log p)] + Y_0(p)$, where

$$\begin{aligned} Y_0(p) & = \log(1 - \zeta(p)) - (1 + \varepsilon(2p)) \times \\ & [1 - \zeta(p) - \Delta(2p) + \delta(2p) \log(1 + \varepsilon(2p))] + \delta(2p)(1 + \varepsilon(2p)) \stackrel{def}{=} \\ & \log g(p) + Y_0(p), \quad Y_2(p) \leq Y_0(p) \leq Y_1(p), \end{aligned}$$

$$Y_1(p) \stackrel{def}{=} \Delta(2p) + \delta(2p) + (1 + \varepsilon_+(2p)) \cdot \delta(p) \log 2 / (1 + \delta(p) \log 2) +$$

$$\varepsilon_+(2p)[\Delta(2p) + \delta(2p)], \quad (4.16d)$$

$$Y_2(p) \stackrel{def}{=} \Delta(2p) + \delta(2p) + \varepsilon_-(2p)[\Delta(2p) + \delta(2p)], \quad (4.16e)$$

where the function $p \rightarrow Y_1(p)$, $p \in [P_1, \infty)$, $P_1 = 10^6$ is monotonically decreasing and

$$\exp(Y_1(P_1)) < 1.442; \quad Y_1(p) \downarrow 0, \quad p \rightarrow \infty; \quad \lim_{p \rightarrow \infty} Y_2(p) = 0. \quad (4.17)$$

7. Upper bound for $L(p)$. We assume in this section that $p \geq P_0 = 700$. Using the well - known Stirling's formula, we obtain for the values $p \geq P_0$:

$$e \cdot L(p) - 1.5 = \sum_{n=3}^{\infty} b_1(n-1, p) \leq \int_2^{\infty} b_1(x, p) dx + \sup_{x \geq 3} b_1(x, p) \leq$$

$$(2\pi)^{-1/2} \exp(p \cdot X(p)) + (2\pi)^{-1/2} \int_2^\infty \exp(V(x, p)) \, dx.$$

Splitting the last integral into three parts so that

$$J(p) \stackrel{\text{def}}{=} \int_2^\infty \exp(V(x, p)) \, dx = J_1 + J_2 + J_3, \quad J_j = J_j(p), \quad j = 1, 2, 3,$$

where

$$J_1(p) = \int_2^{M-\sqrt{p}} \exp(V(x, p)) \, dx, \quad J_2 = \int_{M-\sqrt{p}}^{M+\sqrt{p}} \exp(V(x, p)) \, dx,$$

$$J_3 = \int_{M+\sqrt{p}}^\infty \exp(V(x, p)) \, dx,$$

we obtain for the integral J_2 , taking into account the inequalities $M_- < M < M_+$ and inequality: $x \in [M - \sqrt{p}, M + \sqrt{p}] \Rightarrow$

$$V(x, p) \leq pX(p) - 0.5(x - M)^2 \cdot (p^2 M_+^{-2}) <$$

$$pX(p) - 0.5(x - M)^2 \cdot p \cdot M_+^{-2}(p) :$$

the following:

$$J_2 \leq \exp(p \cdot X(p)) \cdot \int_{M-\sqrt{p}}^{M+\sqrt{p}} \exp\left(-0.5p(x - M)^2 M_+^{-2}\right) \, dx <$$

$$\exp(p \cdot X(p)) \cdot \int_{-\infty}^\infty \exp\left(-0.5p(x - M(p))^2 M_+^{-2}\right) \, dx =$$

$$\sqrt{2\pi} \exp(p \cdot X(p)) M_+ / \sqrt{p} \leq \exp(p \cdot X(p)) \cdot \Psi_1(p),$$

where

$$\Psi_1(p) = \sqrt{2\pi p} \cdot [1 + \Delta + C_{14}\Delta^2] / \log p.$$

Now we estimate the integral J_3 . For the values $x \geq M + \sqrt{p}$ the following inequalities are valid:

$$V(x, p) \leq pX(p) - 0.5 \cdot (2p) \cdot (p/M_+^2(p)) \leq pX(p) -$$

$$\log^2 p \cdot (1 + \Delta + C_{14}\Delta^2)^{-2};$$

$$dV(x, p)/dx \leq -p/M_+^2(p) [x - M(p) - \sqrt{p}];$$

therefore $J_3 \leq \exp(p \cdot X(p)) \cdot \Psi_2(p)$, where $\Psi_2(p) =$

$$\exp\left(-\log^2 p \cdot (1 + \Delta(p) + C_{14}\Delta^2(p))^{-2}\right) \times$$

$$\begin{aligned} & \int_{M+\sqrt{p}}^{\infty} \exp\left(-p M_+^{-2}(x - M - \sqrt{2p})\right) dx = \\ & \exp\left(-\log^2 p \cdot (1 + \Delta(p) + C_{14}\Delta^2(p))^{-2}\right) \times \\ & p \cdot \left(1 + \Delta + C_{14}\Delta^2\right)^2 \cdot \log^{-2}(p) \end{aligned}$$

Analogously, we find the upper estimate for J_1 .

Thus, $L(p) < e^{-1} \cdot \exp(p \cdot X(p)) \times$

$$\begin{aligned} & \left[1.5 \exp(-p \cdot X(p)) + (2\pi)^{-1/2} + \Psi_1(p) + 2(2\pi)^{-1/2} \Psi_2(p)\right] = \\ & \exp(p \cdot X(p)) \cdot \Psi_3^p(p), \end{aligned} \tag{4.18a}$$

where we find by direct calculations: $\Psi_3(P_0) \leq 1.00826$ and at $p \geq P_0$

$$\Psi_3(p) \downarrow 1, \quad p \rightarrow \infty; \quad \Psi_3(p) \leq 1 + C_{18} \log p/p. \tag{4.18b}$$

9. Lower bound for $L(p)$. Denote $q = p - 1/2$. using Sonin's estimate for factorials, we obtain:

$$\begin{aligned} eL(p) & \geq \sum_{n=4}^{\infty} b_1(n-1, p) = \sum_{n=3}^{\infty} b_1(n, p) \geq \\ & \int_4^{\infty} b_1(x, p) dx \geq (2\pi)^{-1/2} \exp(-1/12) \int_4^{\infty} \exp(V(x, q)) dx \geq \\ & (2\pi)^{-1/2} \exp(-1/12) \int_{M(q)}^{M(q)+\sqrt{q}} \exp(V(x, q)) dx. \end{aligned}$$

Since the following implication holds: $x \in [M(q), M(q) + \sqrt{q}] \Rightarrow$

$$V(x, q) \geq q X(q) - 0.5(x - M(q))^2 q M_-^{-2}(q),$$

we have:

$$\begin{aligned} eL(p) & \geq (2\pi)^{-1/2} \exp(-1/12) \exp(qX(q)) \times \\ & \int_{M(q)}^{M(q)+\sqrt{q}} \exp\left[-0.5q M_-^{-2}(q) (x - M(q))^2\right] dx \geq M_-(q) \times \\ & 0.5 \exp(-1/12) \sqrt{q} \exp(qX(q)) \left[1 - \exp\left(-q^2/M_-^2\right)\right] = \\ & e \cdot \exp(p \cdot X(p)) \cdot \Psi_4^p(p), \end{aligned}$$

where

$$\Psi_4(p) \downarrow 1, \quad p \rightarrow \infty; \quad \Psi_4(p) \geq 1 + C_{19} \log p/p.$$

Thus,

$$\exp(p \cdot X(p)) \cdot \Psi_4^p(p) \leq L(p) \leq \exp(p \cdot X(p)) \cdot \Psi_3^p(p), \quad (4.19a)$$

$$\Psi_3(p) \leq 1 + C_{19} \log p/p, \quad \Psi_4(p) \geq 1 + C_{20} \log p/p, \quad (4.19b)$$

$$\Psi_3(p) \downarrow 1, p \rightarrow \infty; \Psi_3(P_0) \leq 1.00826. \quad (4.19c)$$

10. Upper and lower bounds for $K(p)$ are obtained analogously to the upper bound for $L(p)$, but we assume in this section that $p \geq P_1 = 10^6$. In brief,

$$\sum_{k=0}^{\infty} \frac{4^{-k}}{k! (n+k)!} < \frac{1}{n!} \sum_{k=0}^{\infty} \frac{4^{-k}}{k!} = \frac{\sqrt[4]{e}}{n!} < \frac{1.285}{n!},$$

hence

$$K(p) < 2e^{-3/4} \sum_{n=1}^{\infty} \frac{n^p 2^{-n}}{n!} \asymp 2\sqrt[4]{e} \cdot B_2(p; 1/2).$$

Further, we conclude, using the Stirling estimate for factorials again, that:

$$0.5 e^{3/4} K(p) = \sum_{n=1}^{\infty} b_2(n, p) \leq \int_1^{\infty} b_2(x, p) dx + \sup_{x \geq 2} b_2(x, p) \leq$$

$$(2\pi)^{-1/2} \exp(p \cdot Y(p)) + (2\pi)^{-1/2} \int_1^{\infty} \exp(W(x, p)) dx.$$

Again, we split the last integral:

$$\begin{aligned} I_4 &\stackrel{\text{def}}{=} \int_2^{\infty} \exp(W(x, p)) dx = \\ &\left(\int_2^{N(p)-\sqrt{p}} + \int_{N(p)-\sqrt{p}}^{N(p)+\sqrt{p}} + \int_{N(p)+\sqrt{p}}^{\infty} \right) \exp(W(x, p)) dx = \\ I_5 + I_6 + I_7. \end{aligned}$$

As long as at $x > N(p) + \sqrt{p} \Rightarrow$

$$W(x, p) \leq pY(p) - 0.5p^2N_+^2(p) = pY(p) - 0.5 \log^2 p \cdot (1 + \varepsilon_+(2p))^{-2},$$

$$dW/dx \leq -pN_+^{-2}(x - N - \sqrt{p}),$$

we obtain:

$$I_7 \leq \exp(pY(p)) \cdot p \log^{-2} p (1 + \varepsilon_+(2p))^2 \times$$

$$\exp\left(-0.5 \log^2 p (1 + \varepsilon_+(2p))^{-2}\right).$$

Further, if $x \in [N(p) - \sqrt{p}, N(p) + \sqrt{p}]$ then

$$W(x, p) \leq pY(p) - 0.5pN_+^{-2}(p) \cdot (x - N(p))^2.$$

Therefore,

$$\begin{aligned}
I_6 &\leq \exp(pY(p)) \cdot \sqrt{p} (1 + \varepsilon_+(2p)) / \log(2p) \\
\text{and } K(p) &\leq 2e^{-3/4} \exp(pY(p)) \times \\
&\quad \left[(2\pi)^{-1/2} + \sqrt{p}(1 + \varepsilon_+(2p)) / \log(2p) + 2(2\pi)^{-1/2} p \log^{-2} p \right] \times \\
&\quad \left[(1 + \varepsilon_+(2p))^2 \cdot \exp(-0.5 \log^2 p (1 + \varepsilon_+(2p))^{-2}) \right]. \tag{4.20}
\end{aligned}$$

Lower bound for $K(p)$. We obtain: $0.5 e K(p) >$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^p 2^{-n} / n! &= B_2(p; 1/2) > \exp(-1/12) (2\pi)^{-1/2} \exp(qY(q)) \times \\
&\quad \int_{N(q)}^{N(q)+\sqrt{q}} \exp \left[-0.5 q N_-^2(q) (x - N(q))^2 \right] dx \geq \\
&\quad \exp(-1/12) \sqrt{\pi/2} q^{-1/2} N_-(q) \left(1 - \exp \left(-q^2 / N_-^2(q) \right) \right) / \log(2q) = \\
&\quad \exp(-1/12) \sqrt{\pi/2} \exp(qY(q)) \sqrt{q} (1 + \varepsilon_-(2q)) \times \\
&\quad \left(1 - \exp \left(-q^2 / N_-^2(q) \right) \right) / \log(2q).
\end{aligned}$$

Further estimations are similar to the estimation of $L(p)$ and can be omitted. As a result,

$$\exp(p \cdot Y(p)) \cdot \Psi_6^p(p) \leq K(p) \leq \exp(p \cdot Y(p)) \cdot \Psi_5^p(p), \tag{4.21a}$$

where at $p \geq P_1$

$$\Psi_5(p) \leq 1 + C_{21} \log p/p, \quad \Psi_6(p) \geq 1 + C_{22} \log p/p; \tag{4.21b}$$

$$\Psi_5(p) \downarrow 1, \quad p \rightarrow \infty; \quad \Psi_5(P_1) \leq 1.000833. \tag{4.21c}$$

11. For exact computations, we need to estimate the derivatives of our functions $L(p), K(p)$. We show here the estimation of derivatives $L^{(m)}(p), m = 1, 2, \dots$. Namely, $e \cdot L^{(m)}(p) =$

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{(n-1)^p \log^m(n-1)}{n!} &\leq \sum_{n=3}^{\infty} \frac{(n-1)^p}{(n-1)!} \cdot \frac{\log^m n}{n} < \\
&\quad \sum_{n=2}^{\infty} \frac{n^p}{n!} \cdot \left(\sup_{n \geq 3} \frac{\log^m n}{n} \right) = \\
&\quad \left(\frac{m}{e} \right)^m \cdot \sum_{n=2}^{\infty} \frac{n^p}{n!} \asymp \left(\frac{m}{e} \right)^m \cdot (eB_1(p) - 1). \tag{4.22}
\end{aligned}$$

We estimate the derivative $K^{(m)}(p), m = 1, 2, \dots$ in an analogous way.

It follows from this estimation that the functions $L(p)$ and $K(p)$ are infinitely differentiable at the interval $p \in (4, \infty)$. As long as $L(4 - 0) = L(4 + 0) = 4$, $K(4 - 0) = K(4 + 0) = 4$, both functions $K(\cdot)$, $L(\cdot)$ are continuous in the semiclosed interval $[2, \infty)$. However

$$\frac{dK}{dp}(4 - 0) = \frac{dL}{dp}(4 - 0) \approx 3.149195, \quad \frac{dK}{dp}(4 + 0) \approx 3.51934,$$

$$\frac{dL}{dp}(4 + 0) \approx 3.86841,$$

and therefore, both the functions $K(\cdot)$, $L(\cdot)$ are not continuously differentiable in the set $(2, \infty)$. In the open intervals $(2, 4)$ and $(4, \infty)$ all the functions $L(p), K(p), C(p), S(p)$ are infinitely differentiable (see (22) and [18], [19], [16]).

5 Proof of the probabilistic results.

Proof of theorem 4.1. We find by direct calculations that $G(C_4)/g(C_4) \approx 1.77638$, but for we conclude from (4.17) that for the values $p \geq P_0 = 700$

$$G(p)/g(p) \leq 1.00826 \cdot 1.75913 = 1.77366,$$

hence

$$\operatorname{argmax}_{p \in [4, \infty)} G(p)/g(p) \in [4, 700].$$

We obtain by direct calculations using known numerical methods and by means of computer:

$$\max_{p \in [4, 700]} G(p)/g(p) = G(C_4)/g(C_4) \approx 1.77638.$$

Further,

$$\inf_{p \geq 4} G(p)/g(p) = \min \left\{ \min_{p \in [4, 700]} G(p)/g(p), \inf_{p > 700} G(p)/g(p) \right\}.$$

Our computations show that

$$\min_{p \in [4, 700]} G(p)/g(p) \approx 1.332,$$

and it follows from (4.18a), (4.18b), (4.18c) and (4.19) that

$$\inf_{p > 700} G(p)/g(p) = \lim_{p \rightarrow \infty} G(p)/g(p) = 1.$$

Thus,

$$\inf_{p \geq 4} G(p)/g(p) = \lim_{p \rightarrow \infty} G(p)/g(p) = 1.$$

Analogously, $S(C_{10})/g(C_{10}) \approx 1.53572$, but for the values of $p \geq P_1$ the following holds:

$$S(p)/g(p) \leq 1.0008333 \cdot 1.443 < 1.4444.$$

Therefore

$$\operatorname{argmax}_{p \in [4, \infty)} S(p)/g(p) \in [4, 1000000].$$

We have computed the following:

$$\max_{p \in [4, P_1]} S(p)/g(p) = S(C_{10})/g(C_{10}) \approx 1.53572.$$

Further,

$$\inf_{p \geq 4} S(p)/g(p) = \min\left\{\min_{P \in [4, P_1]} S(p)/g(p), \inf_{p \geq P_1} S(p)/g(p)\right\} =$$

$$\lim_{p \rightarrow \infty} S(p)/g(p) = 1.$$

Other assertions of theorem 1 are obtained analogously.

Proof of theorems 4.2 and 4.3. It follows from inequalities (4.19a), (4.19b), (4.19c) and (4.21a), (4.21b), (4.21c) that

$$\exp(X(p)) \cdot (1 + C_{20} \log p/p) \leq G(p) \leq \exp(X(p)) \cdot (1 + C_{19} \log p/p), \quad (4.23a)$$

$$\exp(Y(p)) \cdot (1 + C_{22} \log p/p) \leq S(p) \leq \exp(Y(p)) \cdot (1 + C_{21} \log p/p). \quad (4.23b)$$

Substituting the expression (4.16a) and (4.16b) into equation (4.23a), we obtain, after simple calculation, our assertions (4.9b). We obtain (4.9a) in a similar way.

Finally, substituting expressions (4.16d,e) into (4.23b), we obtain (4.7a), (4.7b).

Proof of Theorem 4.4 Since $\mathbf{E}\theta_{(r)} = 1, r = 0, 1, 2, \dots$, we conclude

$$\mathbf{E}\theta^k = \mathbf{E} \sum_{l=0}^k s(k, l) \theta_{(r)} = \sum_{l=0}^k s(k, l).$$

Equation (4.11a) follows from the binomial formula. The equality (4.10) is proved analogously.

Let us prove (4.11b). Since

$$\sum_{n=2}^{\infty} n^p/n! = e \cdot B_1(p) - 1, \quad p = 1, 2, 3, \dots; \quad \sum_{n=2}^{\infty} 1/n! = e - 2,$$

we conclude for the values $p = 5, 7, 9, \dots$: $e \cdot L(p) =$

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} \frac{(n-1)^p}{n!} &= 1 + \sum_{n=2}^{\infty} (n!)^{-1} \cdot \sum_{k=0}^p (-1)^k \binom{p}{k} n^{p-k} = \\ &= 1 + \left[\sum_{k=1}^p (-1)^k \binom{p}{k} (e B_1(p-k) - 1) \right] - \end{aligned}$$

$$\begin{aligned}
(eB_1(0) - 2) &= 2 + e \cdot \sum_{k=0}^p (-1)^k \binom{p}{k} B_1(p-k) - \sum_{k=0}^p (-1)^k \binom{p}{k} = \\
&= 2 + e \sum_{k=0}^p (-1)^k \binom{p}{k} B_1(p-k).
\end{aligned}$$

6 Some generalization and concluding remark.

Our results allow us to obtain some generalizations of symmetrically distributed random variables $\{\eta(i)\}$, $i = 1, 2, 3, \dots$ on the Hilbert space. Let $(H, \|\cdot\|)$ be a separable Hilbert space with the norm $\|\cdot\|$, $\mathbf{P}(\eta(i) \in H) = 1$, $\forall i = 1, 2, 3, \dots$ $\|\eta\|_p \stackrel{def}{=} \mathbf{E}^{1/p} (\|\eta(i)\|^p) < \infty$, $p \geq 4$,

$$Z(p) = \sup_{\{\eta(i)\}} \sup_n \frac{\|\sum \eta(i)\|_p}{\max(\|\sum \eta(i)\|_2, (\sum \|\eta(i)\|_p^p)^{1/p})}.$$

Utev ([16], [17]) has proved that $Z(p) = S(p)$, $p \geq 4$ (in our notations). Therefore

$$1 = \inf_{p \geq 4} Z(p)/g(p) < \sup_{p \geq 4} Z(p)/g(p) = C_9 \approx 1.53572,$$

$$1 = \inf_{p \geq 15} Z(p)/h(p) < \sup_{p \geq 15} Z(p)/h(p) = C_{11} \approx 1.03734.$$

Moreover, let us examine the following problem. For the values $A, D \in (0, \infty)$, $p = \text{const} \geq 4$ we define

$$Q^p(p, A, D) = \sup_{n \geq 1} \sup_{\{\theta(i)\}} \mathbf{E} \left\| \sum_{i=1}^n \theta(i) \right\|^p,$$

where the interior "sup" is calculated over all the sequences of H -valued independent symmetrically distributed random variables $\{\theta(i)\}$ under the following conditions:

$$\sum_{i=1}^n \mathbf{E} \|\theta(i)\|^2 = D^2, \quad \sum_{i=1}^n \mathbf{E} \|\theta(i)\|^p = A.$$

We denote

$$t = 0.5(A/D^p)^{1/(p-2)},$$

then $t \in (0, 1/2]$. Utev [16], [17] has also proved that

$$Q^p(p, A, D) = (A \cdot D^{-2})^{1/(p-2)} \mathbf{E} |\nu_1 - \nu_2|^p,$$

where ν_1, ν_2 are independent Poisson-distributed random variables with parameter t :

$$\mathbf{P}(\nu_j = k) = \exp(-t) \cdot t^k / k!, \quad k = 0, 1, 2, \dots$$

Hence

$$Q^p(p, A, D) \cdot (D^2/A)^{1/(p-2)} = \mathbf{E}|\nu_1 - \nu_2|^p =$$

$$2e^{-2t} \sum_{n=1}^{\infty} n^p I_n(2t) = e^{-2t} F_2(p; 2).$$

We also denote for $p \geq 4$, $t \in (0, 1/2]$

$$R(p, t) = Q(p, A, D) \cdot (D^2/A)^{1/(p-2)} = 2e^{-2t} \sum_{n=1}^{\infty} n^p I_n(2t),$$

$$u = u(t) = \sup_{p \geq 4} R(p, t)/g(p), \quad T = T(t) = \operatorname{argmax}_{p \geq 4} F(p, t),$$

and using the above-described method, that at $p \rightarrow \infty$, $t = \text{const} \in (0, 1/2]$

$$R(p, t) = [tM(p/t)]^{1-tM(p/t)/p} \left(1 + o\left(\frac{\log p}{p}\right) \right),$$

$$R(p, t) = [p/(e \cdot \log p)] \times$$

$$\left[1 + \frac{\log \log p}{\log p} + \frac{1 + \log t}{\log p} + \frac{\log^2 \log p}{\log^2 p} + o\left(\frac{\log \log p}{\log^2 p}\right) \right],$$

$$R^{2m}(2m, t) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} \sum_{q=0}^{2m-l} \sum_{r=0}^l t^{q+r} s(2m-l, q) s(l, r),$$

$$\inf_{p \geq 4} R(p, t)/g(p) = \lim_{p \rightarrow \infty} R(p, t)/g(p) = 1;$$

TABLE 3

t	T(t)	u(t)
0.45	26.228	1.48566
0.4	32.206	1.43438
0.35	42.120	1.3815
0.3	60.67	1.3281
0.25	102.47	1.2732
0.2	145.96	1.2163

Note that at $t = 1/2 \Rightarrow T(1/2) = C_{10} \approx 22.311$, $u(1/2) = 1.53572$.

Apparently, it is interest to obtain exact constants in the moment inequalities for sums of independent non-negative random variables in the spirit of Refs. [12], [14], [5] etc.

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